	1	5	9	
Third	2 3	6 7	10 11	. . . .
	4	8	12	

All the combinations involving 1, 2, 3, 4 only have been used, and thereby 1 has played *with* each 2, 3, 4 once and twice against each. Since the players are identically involved, *mutatis mutandis*, all the other players will have a similar experience.

361. Proposed by C. E. GITHENS, Ph. D., Wheeling, W. Va.

Find three integral values for $[-10+9\sqrt{(-3)}]^{1/3}+[-10-9\sqrt{(-3)}]^{1/3}$. A solution not involving a cubic is desired.

I. Solution by the PROPOSER.

I. Put $(-10+9\sqrt{-3})^{\frac{1}{3}}+(-10-9\sqrt{-3})^{1/3}=\frac{1}{2}[(-80+72\sqrt{-3})^{1/3}+(-80-72\sqrt{-3})^{1/3}]$.

Let $(-80+72\sqrt{-3})^{1/3}=\sqrt{x}+\sqrt{y}$ and $(-80-72\sqrt{-3})^{1/3}=\sqrt{x}-\sqrt{y}$.

Then, $[6400-(-15552)]^{1/3}=x-y=28$, and, therefore,

$(-80+72\sqrt{-3})^{1/3}=\sqrt{(x+28)}+\sqrt{x}$. Hence, by raising both sides to the third power, $(4x+28)\sqrt{(x+28)}+(4x+84)\sqrt{x}=-80+72\sqrt{-3}$.

Put $4x+28\sqrt{(x+28)}=-80$, and $4x+84\sqrt{x}=72\sqrt{(-3)}$.

Let $4x+28=-80$ or factor of -80; -40, -20, -16, -10, -9, -8, -5, -4, -2, and

$4x+84=72\sqrt{(-3)}$ or factor of 72; 36, 24, 18, 12, 9, 8, 6, 4, 3, 2.

Subtracting, $-56=-20-(36)$.

$\therefore 4x+28=-20$ and $x=-12$.

[1] $\frac{1}{2}[(-80+72\sqrt{-3})^{1/3}+(-80-72\sqrt{-3})^{\frac{1}{3}}]$

$=\frac{1}{2}[(\sqrt{(x+28)}+\sqrt{x})+(\sqrt{(x+28)}-\sqrt{x})]=4$, answer.

II. Similarly with $(-80+72\sqrt{-3})^{\frac{1}{3}}=\sqrt{(x_0+28)}-\sqrt{x_0}$, in which the factors -80 and 24 added to eliminate the $4x$'s produces

$x_0=-27$ and $\frac{1}{2}[(\sqrt{(x_0+28)}-\sqrt{x_0})+\sqrt{(x_0+28)}+\sqrt{x_0}]=-1$, answer.

III. With $-\sqrt{(x_1+28)}+\sqrt{x_1}$ and -16 and 72 as factors as above, $x=-3$ and $\frac{1}{2}[-\sqrt{(x_1+28)}+\sqrt{x_1}+(-\sqrt{(x_1+28)}-\sqrt{x_1})]^{\frac{1}{3}}=-5$, answer.

IV. The root $-\sqrt{(x_2+28)}-x_2$ produces no two factors of -80 and 72 whose difference or sum equals ±56; hence it is not a root, which is as it should be, for the numerical equation is an example of the "irreducible case" in the Cardan solution of a cubic whose equation is $x^3-21x+20=0$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Putting $(-10+9\sqrt{3}.\sqrt{-1})=\rho(\cos\phi+\sin\phi.\sqrt{-1})$, we get $\tan\phi=-\frac{9}{10}\sqrt{3}$. $\rho=\sqrt{343}$, and $(-10+9\sqrt{3}.\sqrt{-1})^{\frac{1}{3}}+(-10-9\sqrt{3}.\sqrt{-1})^{\frac{1}{3}}=2\sqrt{7}\cos\frac{1}{3}\phi$.

$\therefore 4\cos^3\frac{1}{3}\phi-3\cos\phi+\frac{10}{49}\sqrt{7}=0$.

By trial, $\cos\frac{1}{3}\phi=\frac{2}{7}\sqrt{7}$, and dividing the last trinomial by $\cos\frac{1}{3}\phi-\frac{2}{7}\sqrt{7}$, we get $4\cos^2\frac{1}{3}\phi+\frac{8}{7}\sqrt{7}.\cos\frac{1}{3}\phi-\frac{5}{7}=0$; whence $\cos\frac{1}{3}\phi=-\frac{5}{14}\sqrt{7}$, $\cos\frac{1}{3}\phi=\frac{1}{14}\sqrt{7}$.

$\therefore$ The three values required are 4, -5, 1.

Also solved by A. M. Harding.

GEOMETRY.

386. Proposed by DANIEL KRETH, Oxford, Iowa.

Construct the triangle, having given, the vertical angle, the sum of the three sides, and the perpendicular.

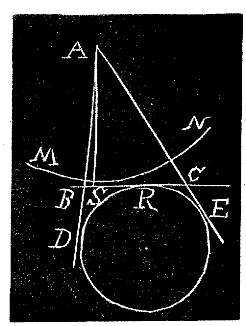
I. Solution by H. PRIME, Boston, Massachusetts.

Let ABC be the required triangle, C the given angle. On AB produced take $BE=BC$. On BA produced take $AF=AC$. Let O be the center of circle ECF. Then we have the angles $FOE=2(BEC+AFC)=ABC+BAC$=supplement of C.

Hence, to construct the triangle, on EF=the given sum of the three sides form the isosceles triangle EOF, making EOF=the supplement of the given vertex angle (or $OEF=OFE$=one half the given angle). About O as center describe the arc EF. Parallel to EF and at a distance from it equal to the given altitude draw a line meeting the arc at C and C'. Draw CA and CB, making the angles $ACF=AEC$ and $BCE=BEC$. ABC is the required triangle.

II. Solution by C. N. SCHMALL, New York City, and A. M. HARDING, University of Arkansas.

Construct an angle A equal to the given vertical angle. Lay off AD and AE each equal to *half* the given sum of the sides. Describe a circle touching these lines in D and E. With A as center and radius equal to the given perpendicular, describe a circle MN. By a well known method draw a line tangent to *both* these circles touching in R and S, respectively, and cutting the sides in B and C. Then ABC is the triangle required.



Proof. $BR=BD$, $CR=CE$.

$\therefore BC=BD+CE$; hence the triangle has the given perimeter. Also, AS is perpendicular to BC; therefore the triangle has the required altitude. Q. E. D.

Also solved by J. Scheffer and A. H. Holmes.